Torsion of shafts with circular symmetry

Introduction

Consider a uniform bar which is subject to a torque T, *e.g.* through the action of two forces F separated by distance d, hence $T = Fd$. Torsion is the resultant twisting of the bar about its longitudinal axis due to the applied torque. For equilibrium there must be an equal magnitude torque applied in the opposite sense at the other end of the bar. This situation is depicted in figure 1.

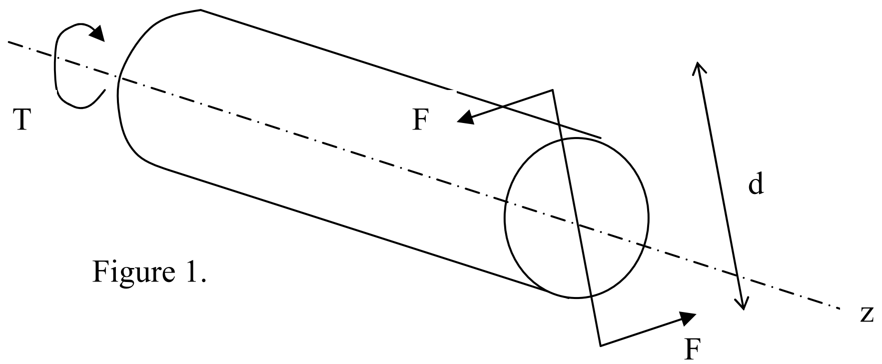


Figure 1.

It can be seen that here the applied force is applied parallel (or tangential) to the end surface of the bar. Therefore the force is termed a *shear* force. (Compare to axial loading where the corresponding normal stress is considered across a plane that is at right angles, or normal, to the applied load).

Shear Stress and Strain (revision)

We adopt a particular notation for considering the application of a shear stress, τ. We typically use two suffixes , e.g. τ_{xy} is a shear stress acting on a $x = const.$ plane and acting in the y direction. For round bars it is typically more convenient to use a cylindrical polar notation (r, θ, z), e.g. $\tau_{z\theta}$ which refers to the shear stress applied to a $z = const.$ plane in the tangential (θ) direction (this is the form of shear stress setup in figure 1 above). A shear stress is considered to be positive when the direction of the stress vector and the normal to the plane are both in the positive or negative senses, e.g. the shear stresses in the left hand element of figure 2 are all positive.

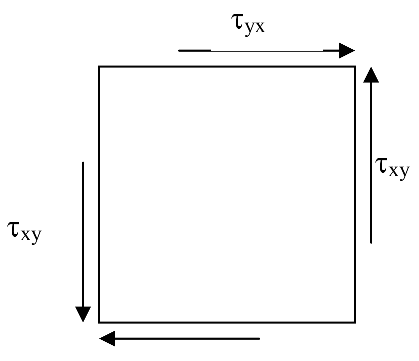


Figure 2.

For equilibrium of the element in figure 2 it is apparent that (take moments about the element's centre) $\tau_{xy} = \tau_{yx}$.

A square element subjected to shear stress as in figure 2 deforms into a diamond shape which exhibits shear strains, see figure 3.

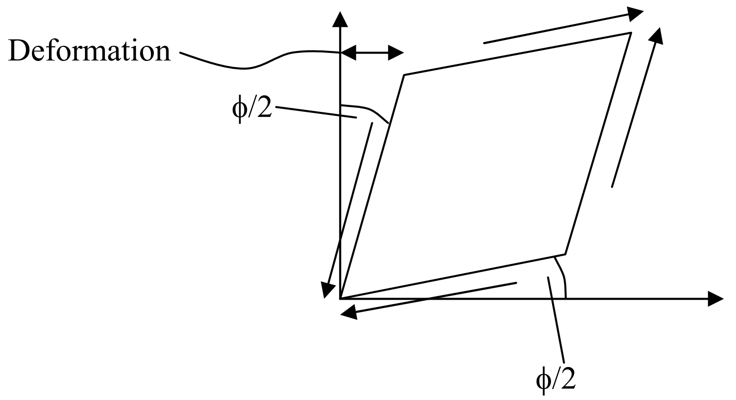


Figure 3

Planes that are initially at right angles are mutually rotated so that the angle between them is reduced by ϕ. We can quantify shear strain by taking the tangent of ϕ. When ϕ is small, $\tan\phi \approx \phi$, so that shear strain can be taken as ϕ (expressed in radians). Shear strain γ is then given by $\gamma = \phi$. Note that $\tan \phi/2$ is equal to the deformation divided by the length of the side of the material element (see figure 3).

γ_{xy} denotes the shear strain in the x-y plane, and can be equivalently written γ_{yx}. Within the elastic limit, shear strain is proportional to shear stress:

$$G = \frac{\tau}{\gamma},$$

where the proportionality constant G is known as the *shear modulus* of the material.

Torsion of members

Consider a shaft of length L that is twisted by angle θ by the application of torque T, e.g. by applying a pair of forces F each at distance $d/2$ from the axis such that $T=Fd$ (see figure 1).

We require:

- the relationship between T and θ,
- the maximum stress.

Note: Often shear force and bending moment will be applied at the same time as torque. Provided the deflections are small, the principle of superposition can be applied, and all loadings treated independently.

The Torque Diagram

To analyse complex torsional problems, we can draw a torque diagram where we consider the torque either applied or extracted from the shaft and the torque carried by the shaft.

To establish a sign convention, we define the torque *T* carried by the shaft as follows: for a shaft lying along the *z*-axis, make an imaginary cut through the shaft. *T* is then the torque that must be applied to the face whose outward normal points in the *z*- direction in order to maintain equilibrium. Therefore, if we consider a torque load on the shaft in the *negative* sense its action will be to *increase* the torque carried by the shaft as we go along the shaft in a positive sense, see figure 4. In this case $T_A = T_1$.

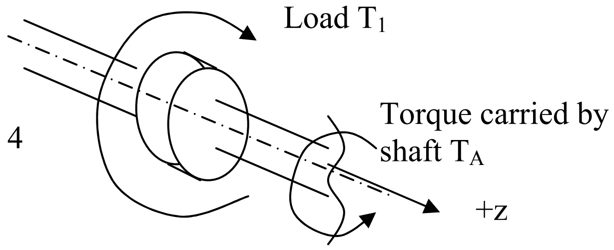


Figure 4

Note:
We will only consider very simple loadings, with torque applied at discrete points along the axis (i.e. in discrete vertical planes). We will *not* consider distributed torsional loads. As a result, any torque diagrams required will be extremely simple. Most problems will be simple enough to solve without the use of a torque diagram.

Example

Consider a shaft (figure 5), driven by a motor via a belt and a pulley, and loaded at several points by other belts, via pulleys. (A familiar sight in old factories and workshops.)

The drive torque T_{IN} acts in the positive sense, while all the loads T_1, T_2, T_3, act in the negative sense, hence the torque carried by the shaft *increases* at each load, and decreases at the drive pulley.

Assuming negligible friction at the end bearings, we must have $T_{IN} = T_1 + T_2 + T_3$.

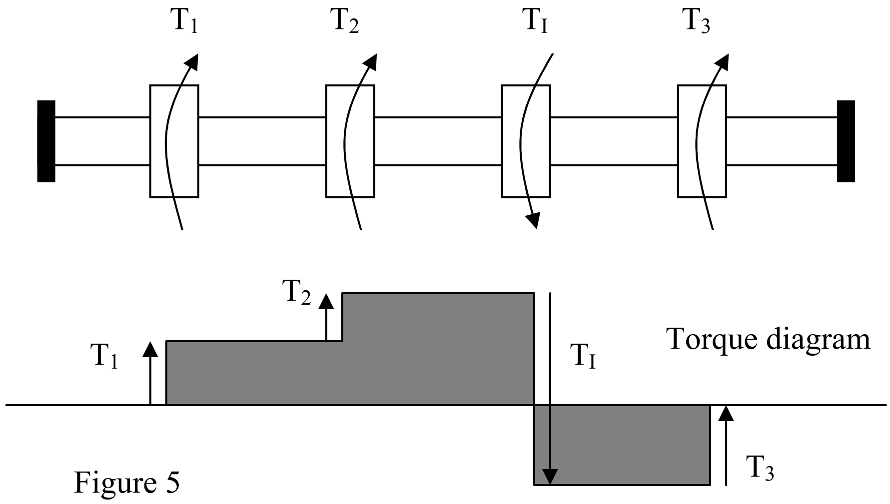


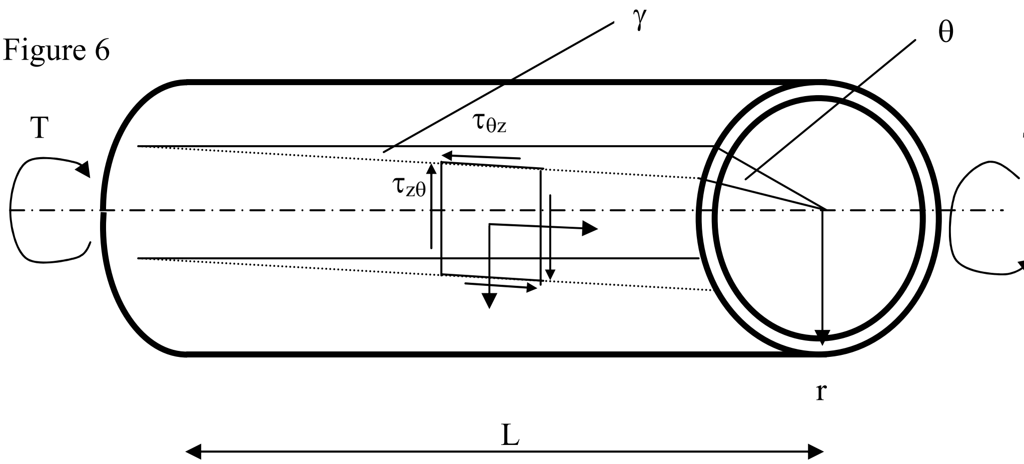
Figure 5

Note that in this case, since drivebelts must be held in tension, each pulley wheel will experience a shear loading. This can be treated separately, by applying superposition, provided transverse deflections are small.

Torsion of a Thin-Walled Cylinder

Consider a thin walled shaft as shown in figure 6. The expression thin-walled can be more rigorously defined as one where the radius, r, to the middle of the shell is far greater than the thickness t of the shell, e.g. $r \geq 10t$. When this condition is satisfied we can assume that the shear stress reaction which occurs in the shell due to the applied torque is constant at a given cross section. The problem is then statically determinant as we can determine the resultant stress purely by consideration of the geometry.

Figure 6



Then we can consider a sector of a cross-section, $d\theta$, where the shear stress $\tau_{z\theta}$ reaction to the torque T acts on a wall area of $trd\theta$ this generates a force:

$dF = \tau_{z\theta} trd\theta$.

This force generates a torque, dT, given by

$dT = r.dF = \tau_{z\theta} tr^2 d\theta$.

Then the total torque is given by

$$T = \int_0^{2\pi} dT = \int_0^{2\pi} \tau_{z\theta} tr^2 d\theta = 2\pi\tau_{z\theta} tr^2 .$$

Or

$$\tau_{z\theta} = \frac{T}{2\pi tr^2} . \qquad (1)$$

We can then consider the amount of deformation which is caused and determine the shear strain, γ. From the figure it is apparent that the deformation is

$\gamma L = r\theta$, and hence

$$\gamma = \frac{r\theta}{L} . \qquad (2)$$

But we also have the equation linking shear stress and shear strain via the shear modulus,

$$G = \frac{\tau}{\gamma}. \qquad (3)$$

Therefore we can combine equations (1), (2) and (3) to give:

$$\tau_{z\theta} = \frac{Gr\theta}{L} = \frac{T}{2\pi t r^2}.$$

Polar Second Moment of Area

The polar second moment of area is defined as the integral across the cross-section of each elemental area multiplied by the square of its distance from the centre of the co-ordinate system. We will only consider circular shafts or tubes.

Polar Second Moment of Area for a Solid Shaft

Examine the circular shaft cross section in figure 7 below.

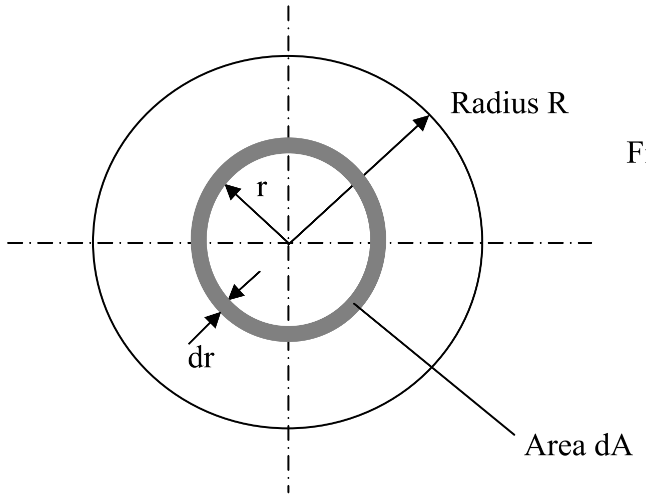


Figure 7

The elemental area is given by $dA = 2\pi r.dr$.
The polar second moment of this area is $r^2.dA = 2\pi r^3.dr$
Then if we integrate over the range from $r = 0$ to $r = R$ we obtain the second moment of area for the whole shaft.

$$Polar\_Second\_Moment\_of\_Area = \int_0^R 2\pi r^3 dr$$

$$Polar\_Second\_Moment\_of\_Area = \frac{\pi R^4}{2}. \qquad (4)$$

Polar Second Moment of Area for a Tube

For a tube of arbitrary dimensions with an internal radius of r_2 and an external radius of r_1.

$$Polar\_Second\_Moment\_of\_Area = \int_{r_2}^{r_1} 2\pi r^3 dr$$

$$Polar\_Second\_Moment\_of\_Area = \frac{\pi\left(r_1^4 - r_2^4\right)}{2}. \qquad (5)$$

Torsion of a Solid Circular Shaft

Preliminaries

We cannot make the assumption that shear stress is constant over a cross section – as we did for the thin tube. However, we can use the symmetry of a shaft – longitudinal, rotational and transverse to allow us to make some deductions about the deformation we expect to occur. Firstly, we will assume that the shaft has a uniform cross-section along its length, it is straight and that the torque is constant along its length. Then we can state the following.

1. A cross-section normal to the longitudinal axis which is planar before application of the torque will remain planar after twisting.
2. A radial line remains radial during twisting.
3. Hence, the deformation occurs from one cross-sectional plane to the next via a rotation about the longitudinal axis.

Stress and Strain as functions of angle of Twist

We can visualise the deformation applied by the torque by considering the shaft to be made up of a set of infinitesimally thin tubes which fit neatly inside each other. A cross section would appear as in figure 8 showing tubes *p* and *q*. From item 2 above radial lines remain radial during twisting, hence the amount of deformation increases linearly with radius.

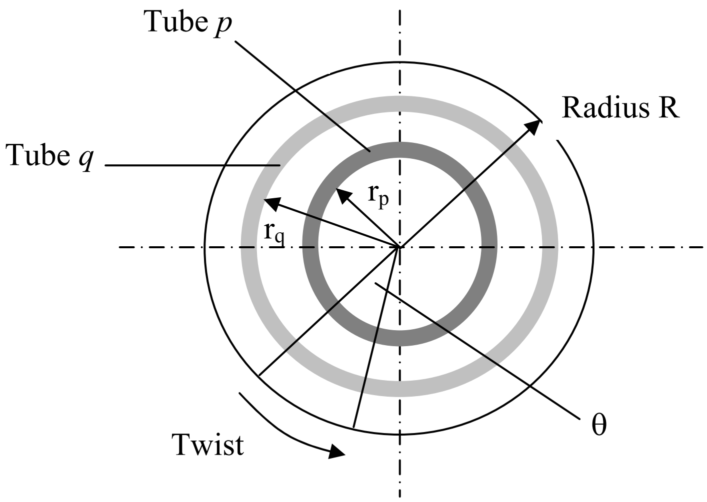


Figure 8

For tubes *p* and *q* we can determine the deflection, δ, they undergo from

$\delta_p = r_p\theta$, $\delta_q = r_q\theta$.

For a shaft of length L and referring to figure 6 we can also write

$\delta_p = \gamma_p L$, $\delta_q = \gamma_q L$.

Equating these expressions leads to

$$\frac{\gamma_p}{r_p} = \frac{\gamma_q}{r_q} = \frac{\theta}{L} = const.$$

This implies that the shear strain increases with radius from 0 at the shaft centre to a maximum value (governed by the shaft length) at the outer radius. Therefore in general we can write:

$$\frac{\gamma}{r} = \frac{\theta}{L}. \qquad (6)$$

We also know that $G = \tau/\gamma$, hence

$$\frac{\tau}{r}=\frac{G\theta}{L}. \qquad (7)$$

Hence the shear stress also varies linearly from 0 at the shaft centre to a maximum value at the outer radius and is a constant for any one of the infinitesimally thin tubes we have visualised.

Torque as a function of Shear Stress

Consider the equilibrium of a cross section. The sum of the torque reactions from internal shear stress in each of the infinitesimally thin tubes must equal the applied torque.
The area of each thin tube is given by $2\pi.r.\delta r$
The force reaction from this tube is then $\tau.2\pi.r.\delta r$
The torque reaction from this tube is $\tau.2\pi.r^2.\delta r$
Then the torque reaction for the whole cross-section is given by:

$$T=\int_o^R \tau.2\pi.r^2 dr. \qquad (8)$$

Substituting for τ from equation 7 into equation 8 gives:

$$T=\int_o^R \frac{G\theta}{L}2\pi.r^3 dr=\frac{G\theta}{L}\int_o^R 2\pi.r^3 dr. \qquad (9)$$

The integral in this expression we recognise as the polar second moment of area for a shaft, J. Therefore from equations 4 and 9

$$T=\frac{G\theta}{L}J,$$

or $\frac{T}{J}=\frac{G\theta}{L}$. (10)

We can obtain an expression relating applied torque and shear stress by combining equations 7 and 10.

$$\tau=\frac{Tr}{J}. \qquad (11)$$

The Torsion Relationship

From equations 7 and 10 we can write.

$$\boxed{\frac{T}{J}=\frac{\tau}{r}=\frac{G\theta}{L} \qquad (12)}$$

Where:

T - Applied torque, Nm
J - Polar second moment of area, m^4
τ - Shear stress, Nm^{-2}, at radius r in m
G - Shear modulus, Nm^{-2}
θ - Angle of twist, $radians$, over length L in m

We can also write

$$\frac{T}{GJ}=\frac{\gamma}{r}=\frac{\theta}{L}. \qquad (13)$$

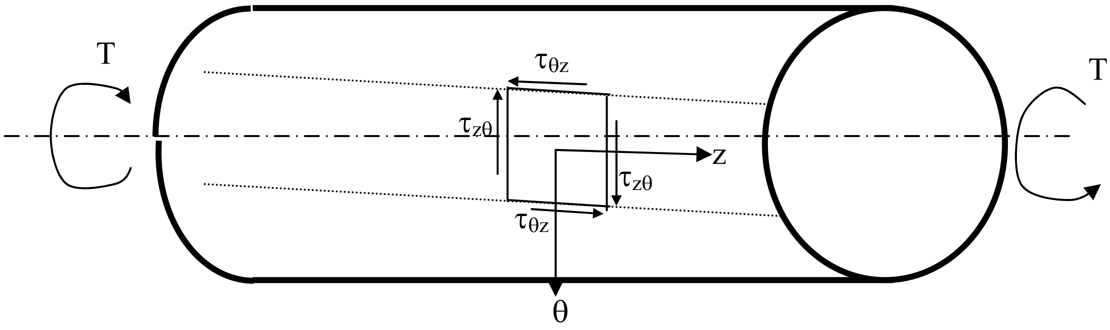
Notes:

1. The analysis here has assumed circular symmetry and does not apply to non-circular shafts (we will only consider circular shafts this year).
2. The product GJ is known as the torsional stiffness as it relates the applied torque to the twist per unit length (compare EA for axial loading of a strut).

How Shafts Fail in Torsion

The shear stress setup through the reaction to applied torque, $\tau_{z\theta}$, also has a complimentary shear stress $\tau_{\theta z}$, see figure 9.

Figure 9



Hence we can construct a Mohr's circle, figure 10, for this state where we have only shear stresses present and no direct stresses.

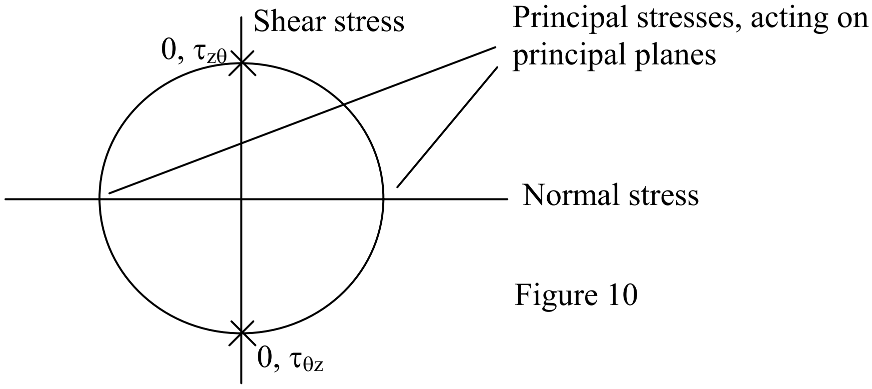


Figure 10

As the principal planes are at 90° to the shear stress planes in Mohr's circle, they will lie at half this angle in the material. Hence the principal stress state is given in figure 11 below, noting that one normal stress is positive and the other is negative.

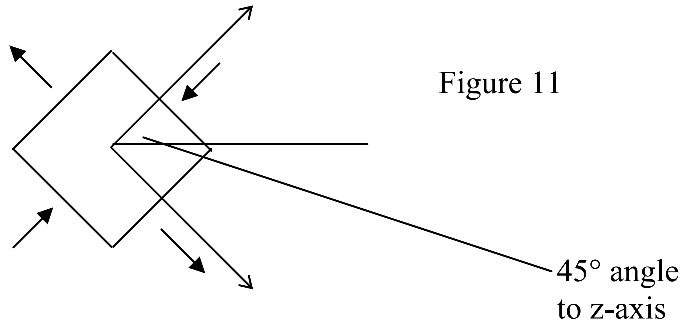


Figure 11

Hence when we apply torque to a shaft, the principal stress system setup is at 45° to the longitudinal axis. We also know that the magnitude of the stress increases with radius, hence, failure will initiate at the outer surface of the shaft (typically at some local imperfection in the material) and propagate through a 45° helix around the shaft – demonstrate this using some chalk!

Torsion of a Hollow Circular Shaft

The analysis for a solid shaft is applicable to a thick tube (of arbitrary dimension) so long as we re-calculate the polar second moment of area accordingly.

Recall that $J = \frac{\pi\left(r_1^4 - r_2^4\right)}{2}$ for a shaft with internal radius r_2 and external radius r_1. The torsion relationship can then be used in the same way as for a solid shaft.

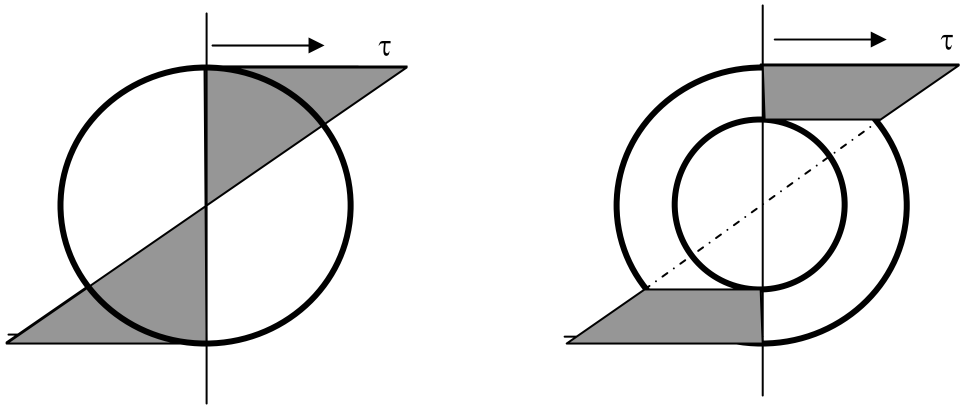
Note that hollow shafts provide a more efficient use of material compared to a solid shaft for two reasons.

i) The magnitude of the area is small at the middle of a shaft, hence the force generated is small.

ii) The magnitude of the shear stress is a linear function of radius and is also small near the shaft centre.

You can evaluate the greater torque carrying capacity of a hollow shaft over a solid shaft of the same material and same weight per unit length by considering the torque as a function of shear stress and geometry from the torsion relationship (see for example 'Mechanics of Engineering Materials', Benham et al, p.107). You will discover that for typical hollow shafts the torque carrying capacity is increased by 44% compared to a solid shaft of the same weight. However, remember that hollow shafts carry increased manufacturing costs.

Comparison of shear stress distributions in a solid and a hollow shaft.



Torsion of Concentric Shafts in Series

Consider the system of shafts shown in figure 12 below.

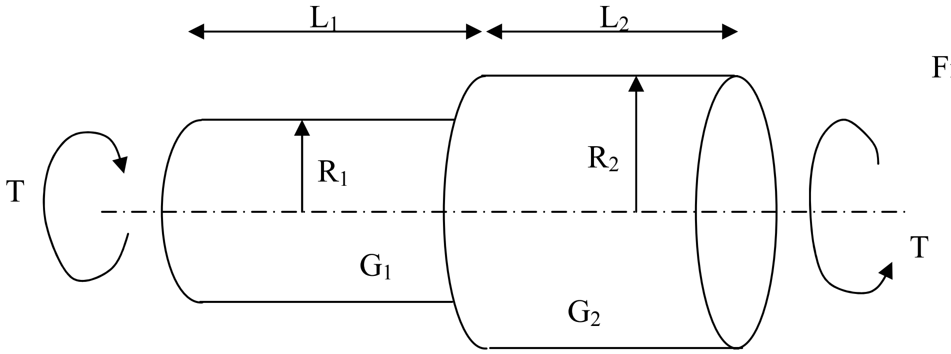


Figure 12

Equilibrium:

$T = T_1 = T_2$.

Geometry of Deformation:

$\theta = \theta_1 + \theta_2$.

Stress-Strain Relationship (Torsion equation):

$$\theta_1 = \frac{TL_1}{G_1 J_1}, \ \theta_2 = \frac{TL_2}{G_2 J_2}.$$

Therefore:

$$\theta = T\left(\frac{L_1}{G_1 J_1} + \frac{L_2}{G_2 J_2}\right).$$

We can also generalise to the case of n shafts in series, where:

$$\theta = T\sum_{i=1}^{n}\frac{L_i}{G_i J_i}. \tag{14}$$

The angle of twist, shear strain and shear stress for each component of the shaft can be found from the torsion equation.

Torsion of Concentric Shafts in Parallel

Consider the system of shafts shown in figure 13 below.

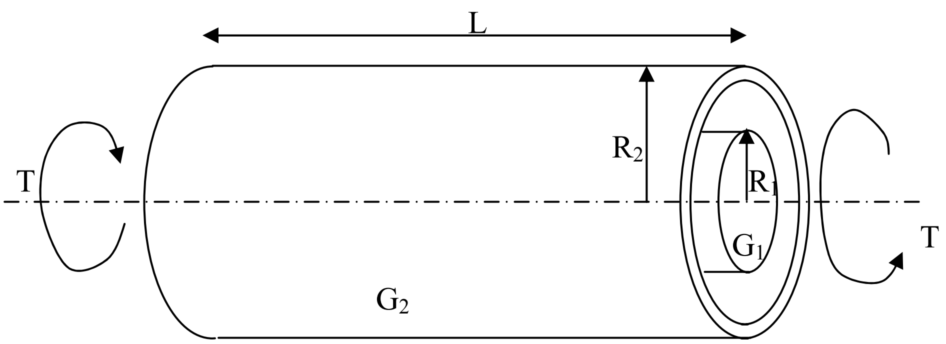


Figure 13

Equilibrium:

$T = T_1 + T_2$.

Geometry of Deformation:

$\theta = \theta_1 = \theta_2$.

Stress-Strain Relationship (Torsion equation):

$$T_1 = \frac{G_1 J_1 \theta}{L},\ T_2 = \frac{G_2 J_2 \theta}{L}. \qquad (15)$$

Therefore:

$$T = \frac{\theta}{L}\left(G_1 J_1 + G_2 J_2\right),$$

or

$$\theta = \frac{TL}{G_1 J_1 + G_2 J_2}.$$

We can also generalise to the case of n shafts in parallel, where:

$$\theta = \frac{TL}{\sum_{i=1}^{n} G_i J_i}. \qquad (16)$$

We can then determine the torque carried by each shaft, equations 15, and the shear stress and shear strain from the torsion equation. (Note the text in 'Mechanics of Engineering Materials', Benham et al, p.110 is confusing as it neglects the suffixes for the torque reacted in shafts 1 and 2 when calculating the shear stress and further assumes that the shear modulus is the same for both shafts. A better treatment is given in 'Strength of Materials' by G.Seed on p.35-36).

Torque Applied to a Shaft with Fixed Ends (Corollary of Shafts in Parallel)

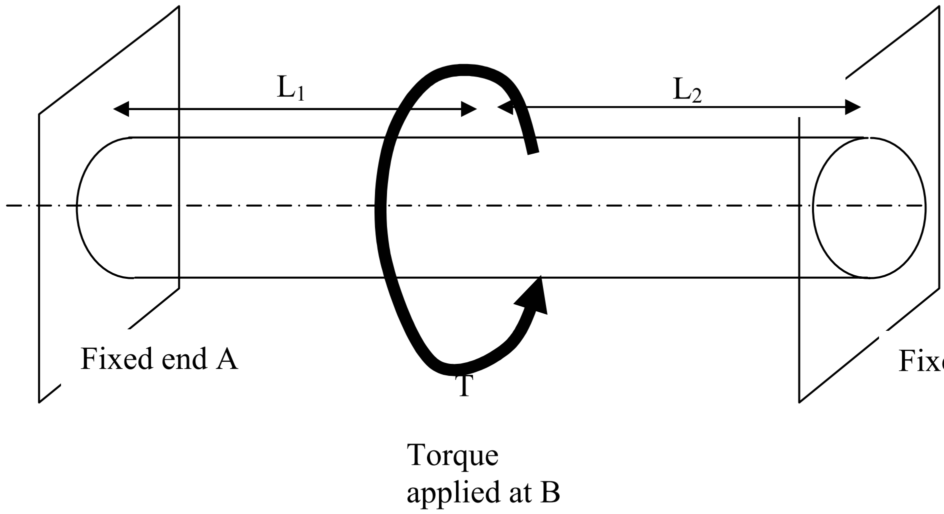


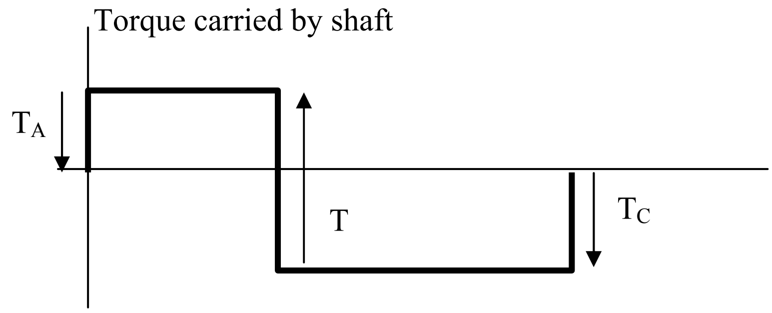
Figure 14

Given a shaft fixed at both ends, A and C, and a torque applied at some point B (+ve) – what occurs? The applied torque must be split over the two sections of the shaft AB and BC. Also the angle of twist for each section of the shaft must be equal in magnitude and opposite in sense. Therefore in many ways this situation is analogous to the two shafts in parallel problem dealt with previously.

We can understand the sign of the torque carried by the shaft by considering FBDs of shaft sections. It is clear that there must be some torque reactions at point A and C which you would expect to be acting in a clockwise manner (i.e. –ve).

If we apply a cut in section AB, the torque carried by the shaft to maintain equilibrium will be $T_{AB} = T_A$. T_{AB} must be anticlockwise and hence +ve.

At point B we apply a +ve torque to the shaft T which will reduce the torque carried over the section BC. Hence $T_{BC} = T_A - T$. This must be –ve as $|T| > |T_A|$. Hence the torque diagram appears as below.



Furthermore the equations we derived for angle of twist for shafts in parallel apply to this situation except we must remember that the shaft lengths AB and BC don't need to be equal. Hence

$$\theta = \frac{T}{\left(\frac{G_1 J_1}{L_1} + \frac{G_2 J_2}{L_2}\right)}.$$

Torsion Effects when adding an Outer Annulus to a Pre-loaded Shaft

This has not been covered in 2000/2001.

Torsion Examples

Example 1.

A solid steel shaft (G = 82 GPa) of circular cross-section, length = 0.5 m, diameter = 20 mm, is twisted about its axis of symmetry by applying a torque of 72Nm. Calculate:

a) The maximum shear stress
b) The angle of twist (degrees)

The torsion equation gives us

$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$, and $J = \frac{\pi r^4}{2}$ for a solid cylinder.

Answer.

a) The maximum shear stress will occur at the outer edge of the cylinder, i.e. at r = 10 mm. Using:

$$\tau_{\max} = \frac{Tr_{\max}}{J} = \frac{72 \times 0.01}{\left(\frac{\pi(0.01)^4}{2}\right)} = 45.83MPa.$$

b) The angle of twist is given by:

$\theta = \frac{TL}{GJ}$, and GJ is the torsional stiffness of the shaft.

$$\theta = \frac{TL}{GJ} = \frac{72 \times 0.5}{82 \times 10^9 \times J} = 0.0279 \text{ radians.}$$

$$\theta = 0.0279 \times \frac{180}{\pi} = 1.60^\circ.$$

Example 2.

The steel shaft from question 1 is stiffened by shrinking on a brass sleeve (G = 38 GPa) of the same length and outer diameter of 30 mm. Assuming this sleeve is rigidly attached over its entire length, calculate:

a) The angle of twist (degrees).
b) The maximum shear stress and shear strain in the brass and steel components.

Answer

a) The shafts act in parallel with the torque split between the two components, and both shafts undergo the same angle of twist per unit length.
Using part of the torque equation:

$$T = \frac{GJ\theta}{L}.$$

Then

$$T = T_{st} + T_{br} = \frac{\theta}{L}\left(G_{st}J_{st} + G_{br}J_{br}\right),$$

or

$$\theta = \frac{TL}{\left(G_{st}J_{st} + G_{br}J_{br}\right)} = \frac{72 \times 0.5}{\left(8.2 \times 10^{10} \times 1.57 \times 10^{-8} + 3.8 \times 10^9 \times 6.38 \times 10^{-8}\right)} = 0.0097 \text{ radians.}$$

Converting to degrees we obtain a twist of 0.55°.

b) To calculate the shear stress we can use:

$\tau = Gr\frac{\theta}{L}$, where we can take the twist per unit length from the above equation as 0.0194 radians/m.
Hence:

$$\tau_{st} = G_{st} r_{\max\_st} \frac{\theta}{L} = 82 \times 10^9 \times 0.01 \times 0.0194 = 15.9 \text{ MPa.}$$

$$\tau_{br} = G_{br} r_{\max\_br} \frac{\theta}{L} = 38 \times 10^9 \times 0.015 \times 0.0194 = 11.1 \text{ MPa.}$$

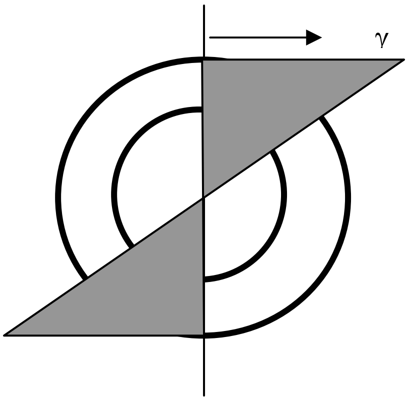
Then for shear strain we have:

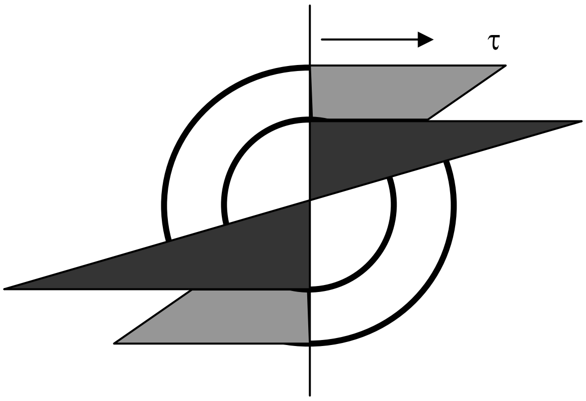
$\gamma = r\frac{\theta}{L}$, and

$$\gamma_{st} = r_{\max\_st} \frac{\theta}{L} = 0.01 \times 0.0194 = 194 \text{ microstrain.}$$

$$\gamma_{br} = r_{\max\_br} \frac{\theta}{L} = 0.015 \times 0.0194 = 291 \text{ microstrain.}$$

Note that the shear strain for the brass is larger than for the steel (as the maximum radius is larger), but that the shear stress for the brass is less than that for the steel as the shear modulus for steel is much larger than that for brass.





Example 3.

A driveshaft must deliver 200 hp at 10,000 rpm to a gearbox (1 hp = 745.7 W). The maximum shear stress in the shaft must not exceed 100 MPa.

a) Calculate the minimum shaft diameter for a solid cylindrical shaft.
b) How much weight can be saved if we use a hollow shaft with $r_2/r_1 = 3/2$

Answer.
a) Given a power requirement and a shaft speed we can calculate the torque transmitted by $P = T\omega$.
Noting that

$$\omega = \frac{revolutions}{\min ute} \times \frac{2\pi(radians)}{1(revolution)} \times \frac{1(\min ute)}{60(\sec onds)} = \frac{radians}{\sec ond}$$

Hence

$$T = \frac{P}{\omega} = \frac{200 \times 745.7}{10000 \times \frac{2\pi}{60}} = 142.4 Nm.$$

Now, we can use part of the torsion equation:

$$\frac{T}{J} = \frac{\tau}{r_o}, r_o = \frac{\tau_{\max} J}{T}$$

But we also know that $J = \frac{\pi . r_o^4}{2}$, hence,

$$r_o = \frac{\tau_{\max} . \pi . r_o^4}{2T}$$

Re-arranging

$$r_o = \left(\frac{2.T}{\pi . \tau_{\max}} \right)^{\frac{1}{3}} = \left(\frac{2 \times 142.4}{\pi . \times 100 \times 10^6} \right)^{\frac{1}{3}} = 9.7 \times 10^{-3} \text{ m}$$

Therefore the shaft diameter needs to be at least 19.4 mm.
Note as the allowable shear stress increases we can decrease the size of the shaft diameter; as the applied torque decreases we can decrease the shaft diameter.

b) For a hollow shaft and the same limiting stress criteria, we assign r_1 as the inner radius, and r_2 as the outer radius. The maximum shear stress will occur at r_2. Hence we can write:

$$r_2 = \frac{\tau_{\max} J_{Hollow}}{T}.$$

Now $J_{Hollow} = \frac{\pi . \left(r_2^4 - r_1^4 \right)}{2}$, and $r_2 = \frac{3r_1}{2}$, or $r_1 = \frac{2r_2}{3}$.

Hence $J_{Hollow} = \dfrac{\pi . \left(r_2^4 - r_2^4 \times \dfrac{16}{81} \right)}{2} = \dfrac{\pi . 0.8025 . r_2^4}{2}$

Therefore,

$$r_2 = \left(\frac{2.T}{\pi . 0.8025 . \tau_{\max}} \right)^{\frac{1}{3}} = \left(\frac{1}{0.8025} \right)^{\frac{1}{3}} . r_o = 1.0761 . r_o$$

Hence the outer diameter is 7.6% larger for the hollow shaft than for the solid shaft to carry the same torque with the same limiting shear stress.

The weight saved will be the ratio of the cross sectional areas for the same shaft lengths.

$$CSA_{Solid} = \pi . r_o^2$$

$$CSA_{Hollow} = \pi \left(r_2^2 - r_1^2 \right) = \pi \left(r_2^2 - r_2^2 \times \frac{4}{9} \right) = 0.555 \pi . r_2^2$$

and using $r_2 = 1.0761 . r_o$,

$$CSA_{Hollow} = 0.6433 \pi . r_o^2.$$

Therefore the hollow shaft is 35.7% lighter than the solid shaft for an increase in radius of 7.6%.